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LIFTING REPRESENTATIONS OF $\mathbb{Z}$-GROUPS

BY

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ABSTRACT

Let K be the kernel of an epimorphism $G \to \mathbb{Z}$, where G is a finitely presented group. If K has infinitely many subgroups of index $2, 3$ or 4, then it has uncountably many. Moreover, if K is the commutator subgroup of a classical knot group G, then any homomorphism from K onto the symmetric group S_2 (resp. $\mathbb{Z}_3$) lifts to a homomorphism onto S_3 (resp. alternating group A_4).

1. Introduction

Let G be a finitely presented group with infinite abelianization. Given an epimorphism χ: $G \to \mathbb{Z}$, we denote its kernel by K. Examples of special interest arise in knot theory; if G is the group $\pi_1(S^3 \setminus k)$ of a knot $k \subset S^3$ and χ is abelianization, then K is the commutator subgroup of G.

In general, K need not be finitely generated. Nevertheless, the Reidemeister–Schreier method [LS77] ensures that it has a group presentation composed of finitely many families of generators $a_j, b_j, \ldots, c_j$ ($j \in \mathbb{Z}$) and relators $r_k, s_k, \ldots, t_k$ ($k \in \mathbb{Z}$) such that any relator in a family can be gotten from any other by shifting all of the indices of its generators by a constant. (Conversely, any group K with such a presentation arises as a kernel χ: $G \to \mathbb{Z}$ for some finitely presented G.) Clearly, K admits a nontrivial $\mathbb{Z}$-action by automorphisms. The action is the restriction to K of conjugation in G by a preimage $x \in \chi^{-1}(1)$; actions

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corresponding to different preimages are related by an inner automorphism of K. For this reason we call K a **finitely presented $\mathbb{Z}$-group** (cf. [Ro96]).

In [SW96] the authors exploited this special structure, showing that for any finite group Σ, the set of representations $\text{Hom}(K, \Sigma)$ has the structure of a *shift of finite type*, a compact 0-dimensional dynamical system completely described by a finite directed graph Γ; in particular, there is a bijection between $\text{Hom}(K, \Sigma)$ and bi-infinite paths in Γ. Techniques of symbolic dynamics can be used to understand $\text{Hom}(K, \Sigma)$. Details are reviewed in §2.

Given any group K, its subgroups of index no greater than r are in finite-to-one correspondence with representations $\rho\colon K \to S_r$, where S_r is the symmetric group on $\{1, \ldots, r\}$. The correspondence can be described by $\rho \mapsto \{g \in K \mid \rho(g)(1) = 1\}$. The preimage of a subgroup of index exactly r consists of $(r-1)!$ transitive representations. By a **transitive** representation we mean a representation ρ such that $\rho(K)$ operates transitively on $\{1, \ldots, r\}$. Note that K contains finitely (resp. countably, uncountably) many subgroups of index r if and only if $\text{Hom}(K, S_r)$ contains finitely (resp. countably, uncountably) many transitive representations. In [SW99] we applied techniques of symbolic dynamics to study $\text{Hom}(K, S_r)$. Under the hypothesis that K *contains an abelian HNN base for G* (see §2) we proved that if K contains infinitely many subgroups of some finite index r, then it contains uncountably many. Our first result is that this dichotomy continues to hold even if the hypothesis is removed, provided that $r < 5$.

THEOREM 3.4 (Dichotomy): *Let K be a finitely presented $\mathbb{Z}$-group. If K contains infinitely many subgroups of index 2, 3 or 4, then K contains uncountably many subgroups of that index.*

The conclusion does not hold if $r > 4$ (see Example 3.4). As an immediate consequence of the above theorem and Corollary 1.3 of [SW99] we obtain

COROLLARY 3.5: *Let K be a finitely presented $\mathbb{Z}$-group. If K contains infinitely many subgroups of index $r = 2, 3$ or 4, then K contains uncountably many subgroups of any index greater than or equal to $r!$.*

When G is a knot group, topology imposes restrictions on the structure of its commutator subgroup K.

THEOREM 4.3: *Assume that K is the commutator subgroup of a knot group. Then (i) any representation from K onto S_2 lifts to a representation onto S_3; (ii) any representation from K onto $\mathbb{Z}_3$ lifts to a representation onto the alternating group A_4.*

The conclusions of Theorem 4.3 do not hold for arbitrary $\mathbb{Z}$-groups (see Example 4.4).

The commutator subgroup of a nontrivial fibered knot is free of rank at least two. Such a group maps onto any symmetric group (and consequently contains subgroups of every index). Does such a conclusion hold for the commutator subgroup of any *nonfibered* knot? We offer a partial answer (see Corollary 3.9).

Much of this paper was inspired by C. Livingston's paper [Li95]. In it he revisits classical results about lifting knot group representations from the perspective of obstruction theory. We found that with the aid of symbolic dynamics many of the techniques extend in a natural way to knot commutator subgroups. From this approach we obtain new insights into the structure of $\mathrm{Hom}(K, \Sigma)$.

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2. Symbolic dynamics and representation shifts

Assume that G is a finitely presented group with epimorphism $\chi: G \to \mathbb{Z}$. Then G can be described as an **HNN extension** $\langle x, B \mid x^{-1}ax = \phi(a),\ \forall a \in U\rangle$. Here $x \in \chi^{-1}(1)$, while B is a finitely generated subgroup of $K = \ker \chi$. The map ϕ is an isomorphism between finitely generated subgroups U, V of B. The subgroup B is an **HNN base**, x is a **stable letter**, ϕ is an **amalgamating map**. Details can be found in [LS77]. It is possible to choose B so that it contains any prescribed finite subset of K (see [Si96], for example).

Conjugation by x induces an automorphism of K. Letting $B_j = x^{-j}Bx^j$, $U_j = x^{-j}Ux^j$ and $V_j = x^{-j}Vx^j$, $j \in \mathbb{Z}$, we can express K as an infinite free product in which each subgroup V_j is amalgamated with U_{j+1}:

$$K = \langle B_j \mid x^{-j}\phi(u)x^j = x^{-j-1}ux^{j+1},\ \forall u \in U,\ \forall j \in \mathbb{Z}\rangle.$$

When Σ is a finite group, the set $\mathrm{Hom}(K, \Sigma)$ can be described by finite directed graph Γ. The vertex set consists of all representations $\rho_0: U \to \Sigma$, a set that is finite since U is finitely generated. If $\bar\rho_0$ is a representation from B to Σ, then we draw a directed edge labeled by $\bar\rho_0$ from the vertex $\rho_0 = \bar\rho_0|_U$ to the vertex $\rho_0' = \bar\rho_0|_V \circ \phi$. Consider any bi-infinite path in Γ given by an edge sequence

$$\cdots \bar\rho_{-2}\bar\rho_{-1}\bar\rho_0\bar\rho_1\bar\rho_2 \cdots.$$

The representations from B_j to Σ defined by $y \mapsto \bar{\rho}_j(x^j y x^{-j})$ have a unique common extension ρ: $K \to \Sigma$. In this way bi-infinite paths of Γ correspond to elements of $\mathrm{Hom}(K, \Sigma)$.

Let E be a finite set (with discrete topology) and give $E^{\mathbb{Z}}$ the product topology. We define the **shift** homeomorphism σ by $(\sigma y)_j = y_{j+1}$ for $y = (y_j) \in E^{\mathbb{Z}}$. The dynamical system $(E^{\mathbb{Z}}, \sigma)$ is called the **full shift** on the symbol set E. Any closed σ-invariant subset is a **subshift**. In a slight abuse of notation, we will use the same symbol σ for the restriction of σ to a subshift, and also for subshifts on different symbol sets. A subshift Y is an n-step **shift of finite type** if there is a subset S of E^{n+1}, the set of **allowed** $(n+1)$**-blocks**, such that Y consists of precisely those $y = (y_j)$ for which $y_j \cdots y_{j+n} \in S$, for all j. (Following the custom in symbolic dynamics and information theory, we write elements of E^n as words $y_1 \cdots y_n$ instead of n-tuples.) A finite directed graph Γ determines a 1-step shift of finite type X_Γ, the **edge shift** of Γ: the symbol set is the edge set E and ee' is an allowed 2-block if the terminal vertex of e is the initial vertex of e'. Thus X_Γ is the set of elements of $E^{\mathbb{Z}}$ that correspond to bi-infinite paths in Γ. The graph constructed in the preceding paragraph presents $\mathrm{Hom}(K, \Sigma)$ as a shift of finite type. As we showed in [SW96], the product topology on the shift space coincides with the compact-open topology on $\mathrm{Hom}(K, \Sigma)$ and the shift map σ is induced by the map σ_x: $\mathrm{Hom}(K, \Sigma) \to \mathrm{Hom}(K, \Sigma)$ given by

$$(\sigma_x \rho)(a) = \rho(x^{-1} a x), \quad \forall a \in K.$$

Two dynamical systems (Y, τ) and (Y', τ') are **topologically conjugate**, or simply **conjugate**, if there is a homeomorphism h: $Y \to Y'$ with $\tau' \circ h = \tau$. Although the graph presenting $\mathrm{Hom}(K, \Sigma)$ depends on the choice of HNN base B and stable letter x, the corresponding shift of finite type is uniquely determined up to conjugacy by G and χ. As in [SW96] we call the pair $(\mathrm{Hom}(K, \Sigma), \sigma_x)$ the **representation shift** of K in Σ, denoted by (Φ_Σ, σ) or more simply Φ_Σ. An algorithm, based on the Reidemeister–Schreier method, for obtaining a graph presenting Φ_Σ from a finite presentation of G is given in [SW96].

Example 2.1: Consider the Baumslag–Solitar group $G = \langle x, a \mid ax = a^2 x \rangle$ together with the epimorphism χ: $G \to \mathbb{Z}$ mapping $x \mapsto 1$ and $a \mapsto 0$. The group G has an HNN decomposition such that $B = U$ is the infinite cyclic subgroup $\langle a \rangle$, $V = \langle a^2 \rangle$ and $\phi(a) = a^2$. The kernel K of χ has presentation $\langle a_j \mid a_j^2 = a_{j+1}\ \forall j \in \mathbb{Z} \rangle$.

In this example $\Phi_{\mathbb{Z}_3}$ has exactly 3 elements: the trivial representation; the representation ρ mapping $a_{2j} \mapsto 1$ and each $a_{2j+1} \mapsto 2$; and the representation

$\sigma\rho$ mapping $a_{2j} \mapsto 2$ and each $a_{2j+1} \mapsto 1$. The graph Γ that describes the representation shift appears in Figure 1. Here we label the vertex ρ_0 by $\rho_0(a)$.

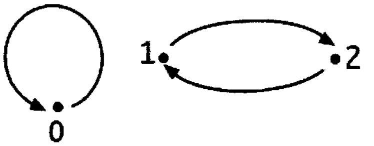

Figure 1. Graph Γ describing $\Phi_{\mathbb{Z}_3}$.

We close this section with some notions from symbolic dynamics that will be needed in the sequel. We refer the reader to [LM95] for additional background.

If Y is a subshift of $E^{\mathbb{Z}}$ and n a positive integer, we may construct a conjugate subshift $Y^{(n)}$ of $(E^n)^{\mathbb{Z}}$, the n**-block presentation** of Y, by sending the sequence $(y_j) \in Y$ to the sequence of overlapping n-blocks

$$\cdots(y_j\cdots y_{j+n-1})(y_{j+1}\cdots y_{j+n})(y_{j+2}\cdots y_{j+n+1})\cdots.$$

If Y is an (at most) n-step shift of finite type, then $Y^{(n)}$ is a 1-step shift of finite type described by a graph in which the edges are allowed $(n+1)$-blocks of Y and the initial and terminal vertices of an edge are its initial and terminal n-blocks. If we replace the HNN base $B = B_0$ in our construction of a graph presenting Φ_Σ by the larger base $B^{(n)} = \langle B_0 \cup B_1 \cup \cdots \cup B_{n-1}\rangle$, we obtain a graph for the n-block presentation.

We may, and will, assume that the graph Γ of a shift of finite type X_Γ has been "pruned" to remove all vertices and edges that do not lie on a bi-infinite path. Then X_Γ is **irreducible** if Γ is strongly connected, that is, there is a path from every vertex to every other vertex. The **irreducible components** of X_Γ are the finite type subshifts corresponding to the maximal strongly connected subgraphs of Γ.

A point y of a subshift has **period** r if $\sigma^r y = y$. We will refer to periodic points of the representation shift Φ_Σ as periodic representations. Periodic orbits of X_Γ correspond to closed paths in Γ. If a point y of a shift of finite type has period at most r, then it is easily seen that its orbit is represented by a simple closed path in the graph of the r-block presentation. The periodic points of X_Γ form a dense subset if and only if X_Γ is the union of its irreducible components. The subshift X_Γ is finite if and only if Γ is a disjoint union of cycles, and uncountable if and only if Γ has a strongly connected subgraph containing more than one cycle.

A **Markov group** is a shift of finite type that is simultaneously a group under an operation that is preserved by the shift map. If E is a group, then the full shift $E^{\mathbb{Z}}$ is a Markov group under coordinatewise multiplication, and so is any finite type subshift of $E^{\mathbb{Z}}$ that is also a subgroup. For example, $\Phi_\Sigma = \mathrm{Hom}(K, \Sigma)$ is a Markov group if Σ is abelian. Theorem 6.3.3 of [Ki98] describes the structure of a Markov group: The irreducible component that contains the identity element is topologically conjugate to a full shift (possibly the trivial shift $\{e\}^{\mathbb{Z}}$), and the entire Markov group is conjugate to the product of this full shift with a finite Markov group (which may also be trivial). It follows that a Markov group has dense periodic point set. It is finite if the full shift factor is trivial, and otherwise every irreducible component is uncountable.

If K is the commutator subgroup of a knot group, then Φ_Σ must be finite for every abelian Σ. For otherwise, the full shift component would contain a nontrivial fixed point ρ. Then $\rho(a) = \sigma\rho(a) = \rho(x^{-1}ax)$, or $\rho(a^{-1}x^{-1}ax) = e$, for all $a \in \rho$. But the knot group is the normal closure of x, and so the elements $a^{-1}x^{-1}ax$ generate K (see [HK78], for example).

3. Lifting representations

Assume that E is an extension of Σ by an abelian group A:

$$0 \to A \to E \xrightarrow{p} \Sigma \to 1.$$

For notational convenience, we identify A with its image $i(A)$, and regard the latter as a multiplicative group.

The group E acts on A by conjugation. Similarly, there is an induced action of Σ on A defined by $a^y := \tilde{y}a\tilde{y}^{-1}$, where $\tilde{y}$ is any preimage of y; in this way we regard A as a Σ-module. If a homomorphism $\rho\colon K \to \Sigma$ lifts to $\tilde{\rho}\colon K \to E$, then A acts by conjugation on the set of liftings.

The homomorphism p induces a continuous mapping $p^*\colon \Phi_E \to \Phi_\Sigma$. Moreover, p^* is **shift-commuting** in the sense that $p^* \circ \sigma = \sigma \circ p^*$. If the extension splits then p^* is onto.

The following proposition is a standard application of the cohomology theory of group extensions (see [Br82], for example).

Proposition 3.1: *Assume that $\rho \in \Phi_\Sigma$ has a lifting $\tilde{\rho} \in \Phi_E$.*

(i) *The set of ρ-***twisted cocycles**,

$$C^1(K; \{A\}) = \{\xi\colon K \to A \mid \xi(xy) = \xi(x)\xi(y)^{\rho(x)}\},$$

corresponds via $\xi \mapsto \rho_\xi$, *where* $\rho_\xi(x) = \xi(x)\tilde{\rho}(x)$, *to the complete set of liftings.*

(ii) *Liftings* $\tilde{\rho}_1, \tilde{\rho}_2$ *are* A*-conjugate if and only if* $\tilde{\rho}_2(\tilde{\rho}_1)^{-1}$ *is a* **coboundary**, *a map of the form* $x \mapsto a^{\rho(x)}a^{-1}$, *for some fixed* $a \in A$.

Proof: (i) Given a derivation ξ, define $\rho_\xi \colon K \to E$ by $\rho_\xi(x) = \xi(x)\tilde{\rho}(x)$. Then

$$\begin{aligned}\rho_\xi(xy) &= \xi(xy)\tilde{\rho}(xy) = \xi(x)\xi(y)^{\rho(x)}\tilde{\rho}(x)\tilde{\rho}(y)\\ &= \xi(x)\tilde{\rho}(x)\xi(y)\tilde{\rho}(y) = \rho_\xi(x)\rho_\xi(y).\end{aligned}$$

Hence ρ_ξ is in Φ_E, and it is a lift of ρ.

Conversely, given a lift $\hat{\rho}$ of ρ, define $\xi \colon K \to A$ by $\xi(x) = \hat{\rho}(x)(\tilde{\rho}(x))^{-1} \in \mathrm{Ker}(p) = A$. Then

$$\begin{aligned}\xi(xy) &= \hat{\rho}(xy)(\tilde{\rho}(xy))^{-1} = \hat{\rho}(x)\hat{\rho}(y)(\tilde{\rho}(y))^{-1}(\tilde{\rho}(x))^{-1}\\ &= \hat{\rho}(x)(\tilde{\rho}(x))^{-1}[\hat{\rho}(y)(\tilde{\rho}(y))^{-1}]^{\rho(x)}\\ &= \xi(x)\xi(y)^{\rho(x)}.\end{aligned}$$

Hence $\xi \in C^1(K; \{A\})$ and $\hat{\rho} = \rho_\xi$. Clearly, distinct ξ give distinct ρ_ξ. The proof of (ii) is equally routine, and we leave it to the reader. ∎

Remark 3.2: Proposition 3.1 implies that the A-conjugacy classes of liftings correspond to elements of the cohomology group $H^1(K; \{A\})$, with coefficients in A twisted by the action of K. If X is a complex with $\pi_1 X \cong K$, then $H^1(K; \{A\})$ is isomorphic to $H^1(X, A)$ (see Proposition 2 of [Li95]).

LEMMA 3.3: *Let* $\rho \in \Phi_\Sigma$ *be a periodic representation with lifting* $\tilde{\rho} \in \Phi_E$. *The preimage under* p^* *of the orbit of* ρ *is a shift of finite type with dense set of periodic points, that is, a disjoint union of irreducible shifts of finite type. The irreducible components are all finite or all uncountable.*

Proof: As in §2, regard G as an HNN extension with finitely generated HNN base B and stable letter $x \in \chi^{-1}(1)$. Then Φ_Σ is described by a graph with edge set $\mathrm{Hom}(B, \Sigma)$, and Φ_E by a graph with edge set $\mathrm{Hom}(B, E)$. Let r be the least period of ρ. Replacing B with the larger HNN base $B^{(r)} = \langle B_0 \cup \cdots \cup B_{r-1}\rangle$ if necessary, we can assume that the orbit O of ρ is represented by a simple cycle in this graph. Then the set $\tilde{O}$ of lifts of O is the finite type subshift of Φ_E described by the subgraph consisting of the edges $\eta \in \mathrm{Hom}(B, E)$ for which $p \circ \eta$ is an edge in the cycle presenting ρ.

For $\xi \in C^1(K;\{A\})$ we define $\sigma\xi\colon K \to A$ by $\sigma\xi(y) = \xi(x^{-1}yx)$. It is easy to check that $\sigma\xi$ is a $\sigma\rho$-twisted cocycle. Since ρ has period r, we obtain an action of $\tau = \sigma^r$ on $C^1(K;\{A\})$. We claim that the pair $(C^1(K;\{A\}),\tau)$ can be viewed as a shift of finite type. The symbol set is the set $C^1(B^{(r)};\{A\})$. An element $\xi \in C^1(K;\{A\})$ can be identified with the sequence of symbols $\xi_j = \tau^j\xi|_{B^{(r)}}$, and $\xi\xi'$ is an allowed 2-block if $\xi|_{V_{r-1}} \circ \phi = \xi'|_U$.

It is straightforward to check that $C^1(K,\{A\})$ is an abelian group under the operation $(\xi+\eta)(y) = \xi(y)\eta(y)$, and that τ respects this addition. Hence $(C^1(K,\{A\}),\tau)$ is a Markov group. By the Kitchens structure theorem cited in the preceding section, it is a disjoint union of finitely many shifts of finite type which are all finite or all uncountable.

By Proposition 3.1, the set $\mathcal{P}$ of all lifts of ρ is the set of products $\rho_\xi = \xi\tilde{\rho}$ with $\xi \in C^1(K,\{A\})$. Now

$$\sigma^r(\xi\tilde{\rho})(y) = (\xi\tilde{\rho})(x^{-r}yx^r) = \tau\xi(y)\tilde{\rho}(y),$$

so we can identify the dynamical system $(\mathcal{P},\sigma^r)$ with the Markov group $(C^1(K,\{A\}),\tau)$. The shift of finite type $\tilde{O}$ is equal to $\mathcal{P} \cup \sigma\mathcal{P} \cup \cdots \cup \sigma^{r-1}\mathcal{P}$. Clearly each irreducible component of $(\mathcal{P},\sigma^r)$ lies in a unique irreducible component of $(\tilde{O},\sigma)$, and $\tilde{O}$ is the disjoint union of these. The components of $\tilde{O}$ are finite or uncountable according as the components of $\mathcal{P}$ are finite or uncountable. ∎

THEOREM 3.4: *Assume that K is a finitely presented $\mathbb{Z}$-group, and $n = 2, 3$ or 4. Then K has either finitely or uncountably many subgroups of index n.*

Proof: It suffices to show that for these n, $\mathrm{Hom}(K,S_n) = \Phi_{S_n}$ contains either finitely or uncountably many transitive representations. We may naturally regard Φ_{S_2} and Φ_{S_3} as subshifts of Φ_{S_4} given by subgraphs of the graph describing Φ_{S_4}.

Since $S_2 \cong \mathbb{Z}_2$ is abelian, Φ_{S_2} is a Markov group, and hence it is either finite or uncountable. Since all nontrivial elements are transitive, the theorem holds for $n = 2$.

If Φ_{S_2} is uncountable, then the irreducible component containing the trivial representation is uncountable. In the graph describing Φ_{S_2}, the edge corresponding to the trivial representation in $\mathrm{Hom}(B,S_2)$ begins and ends at the vertex v corresponding to the trivial representation in $\mathrm{Hom}(U,S_2)$, and there must be another path p from v to itself corresponding to a nontrivial periodic representation ρ. Conjugating ρ with the transpositions (23) and (24) gives representations ρ' in Φ_{S_3} and ρ'' in Φ_{S_4} that correspond to paths p' and p'' from v to itself in the graphs describing those representation shifts. We have $(12) \in \rho(K)$, and so $(13) \in \rho'(K)$ and $(14) \in \rho''(K)$. Freely concatenating p and p' gives uncountably many bi-infinite paths that correspond to transitive representations in Φ_{S_3}; concatenating the paths p, p' and p'' gives uncountably many transitive representations in Φ_{S_4}.

Suppose that Φ_{S_2} is finite but Φ_{S_3} is not. We have a short exact sequence $\mathbb{Z}_3 \cong \langle (123) \rangle \to S_3 \to S_2$ that splits, giving an epimorphism $\Phi_{S_3} \to \Phi_{S_2}$. There must be a periodic representation $\rho \in \Phi_{S_2}$ that has infinitely many lifts in Φ_{S_3}; ρ itself is one of them. Applying Lemma 3.3, we see that the irreducible component containing ρ in the lift of the orbit of ρ to Φ_{S_3} is uncountable. Its graph contains a closed path p corresponding to ρ and another, $\tilde{p}$, corresponding to a lifting $\tilde{\rho} = \xi\rho$ where ξ is a nontrivial element of $C^1(K; \{\mathbb{Z}_3\})$. Then either $\tilde{\rho}(K)$ contains the 3-cycle (123), or $\rho(K)$ contains (12) and $\tilde{\rho}(K)$ contains (13) or (23). There are uncountably many bi-infinite paths in this component that contain both p and $\tilde{p}$ and so correspond to transitive representations into S_3. Conjugation by the transposition (34) fixes ρ but sends $\tilde{\rho}$ to a new periodic representation $\bar{\rho} \in \Phi_{S_4}$, which must therefore be in the same irreducible component of Φ_{S_4} as ρ and $\tilde{\rho}$. By a similar argument, this component contains uncountably many transitive representations into S_4.

Finally, suppose that Φ_{S_3} is finite. Applying Lemma 3.3 to the extension $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \cong \langle (12)(34), (13)(24) \rangle \to S_4 \to S_3$ and to each of the periodic orbits of Φ_{S_3}, we see that Φ_{S_4} is a disjoint union of irreducible components, and thus has dense set of periodic points. Each component is finite or uncountable. If Φ_{S_4} contains infinitely many transitive representations, then so does some irreducible component. We can find a closed path p in the graph of this component corresponding to a periodic transitive representation. Then uncountably many bi-infinite paths in this graph contain p and so correspond to transitive representations. ∎

Corollary 3.5: *Let K be a finitely presented $\mathbb{Z}$-group. If K contains infinitely many subgroups of index $r = 2, 3$ or 4, then K contains uncountably many subgroups of any index greater than or equal to $r!$.*

Proof: Assume that K contains infinitely many subgroups of index $r = 2, 3$ or 4. Theorem 3.4 implies that there are in fact uncountably many. Consequently, K admits uncountably many transitive representations into S_r. It follows that among the kernels of the representations there are uncountably many subgroups of index no greater than $r!$. Theorem 1.2 of [SW99] implies that K contains uncountably many subgroups (not necessarily normal) of any index greater than or equal to $r!$. ∎

It is not difficult to see that the conclusion of Theorem 3.4 holds for arbitrary groups when $r = 2$. However, it does not hold for arbitrary groups when $r = 3, 4$. Nor does it hold for $\mathbb{Z}$-groups when $r = 5$. The assertion for $r = 3$ is established by the following example, a slight modification of an example generously provided by Jim Howie.

Example 3.6: Every index-3 subgroup H of a group G arises as the preimage $\rho^{-1}(\langle(12)\rangle)$, for some transitive representation $\rho\colon K \to S_3$. Moreover, H is normal if and only if the image of ρ is abelian.

Let G be the direct sum of countably many copies of S_3. Coordinate projections are transitive representations, yielding countably many subgroups of index 3. We will show that G has no other subgroups H of index 3.

Observe that H cannot be normal. If it were, then $G/H \cong \mathbb{Z}/3$ would be a quotient of $G/[G,G]$. However, $G/[G,G]$ is a direct sum of countably many copies of $\mathbb{Z}/2$, and any quotient has exponent 2. Hence H arises from a surjective representation $\rho\colon G \to S_3$. We can see that the only such representations are projections composed with automorphisms of S_3. For let $a = (12), b = (123)$, which generate S_3. We regard G as generated by $a_i, b_i,\ i \in \mathbb{N}$, where a_i, b_i are generators of the ith factor of G. Since the image of ρ is not abelian, some b_j must be mapped to an element of order 3. The corresponding generator a_j must be sent to an element of order 2. Then no other generator $a_i, b_i,\ i \neq j$ can be mapped nontrivially, since such generators commute with a_j and b_j in the direct sum.

In a similar way, one can show that the direct sum of countably many copies of A_4 contains countably but not uncountably many subgroups of index 4. Details are left to the reader.

Example 3.6 is notable in another respect. Using a trick of [St84] (see page 273), one can see that G is the kernel of a homomorphism from a finitely presented group onto $\mathbb{Z}^2$. Hence the conclusion of Theorem 3.4 does not hold for finitely presented $\mathbb{Z}^d$-groups, defined in the obvious manner, when d is greater than 1.

We show that the conclusion of Theorem 3.4 does not hold when $r = 5$ by modifying Example 3.2 of [SW99′], an example due to K. H. Kim and F. Roush.

Example 3.7: The alternating group A_5 has presentation $\langle a, b \mid a^2, b^3, (ab)^5 \rangle$. Consider the HNN extension $G = \langle x, B \mid x^{-1}ax = \phi(a), \forall a \in U \rangle$, where

$$B = \langle a, b, a', b' \mid a^2, b^3, (ab)^5, a'^2, b'^3, (a'b')^5, [a, a'], [b, b'], [aa'^{-1}, b'][bb'^{-1}, a'] \rangle,$$

a quotient of the free product of two copies of A_5; U and V are the subgroups generated by a, b and a', b' respectively; and the homomorphism ϕ: $U \to V$ maps $a \mapsto a', b \mapsto b'$. As in §2, let K be the kernel of the epimorphism χ: $G \to \mathbb{Z}$ that sends $x \mapsto 1$ and $a, b, a', b' \mapsto 0$. We will show that the representation shift Φ_{S_5} is countable.

We construct a graph describing Φ_{S_5} as in section 2. A vertex $v \in \mathrm{Hom}(U, S_5)$ is determined by $v(a) = \alpha$, $v(b) = \beta$, so we can identify the vertex set with the set of pairs $(\alpha, \beta) \in S_5 \times S_5$ with $\alpha^2 = \beta^3 = (\alpha\beta)^5 = e$. There is an edge $\bar{\rho}$ from (α, β) to (α', β') if and only if the assignment $\bar{\rho}(a) = \alpha$, $\bar{\rho}(b) = \beta$, $\bar{\rho}(a') = \alpha'$, $\bar{\rho}(b') = \beta'$ defines an element of $\mathrm{Hom}(B, S_5)$. Since A_5 is a simple subgroup of index 2 in S_5, the subgroup generated by α and β is trivial, cyclic of order 2 or A_5. The second case, in fact, cannot occur: β would have to be trivial, thereby forcing $\alpha^2 = \alpha^5 = e$ and so $\alpha = e$ as well.

Note that every vertex admits a self-loop (that is, an edge which begins and ends at the vertex) and an edge from it to the vertex (e, e). As in [SW99′], we claim that there are no other edges in Γ. To see this, suppose that there exists an edge from (e, e) to some other vertex (α, β). From the presentation above, we see that α and β must commute, and hence $\alpha = \beta = e$. Now suppose that there exists an edge from a vertex (α, β) to another (γ, δ), neither of which is (e, e). Again using the presentation we see that $\alpha\gamma^{-1}$ and $\beta\delta^{-1}$ commute with both γ and δ. Since the center of A_5 is trivial, $\alpha = \gamma$ and $\beta = \delta$ and hence the edge is merely a self-loop on (α, β).

It is clear that the graph we have described has countably many bi-infinite paths, and so the representation shift Φ_{S_5} is countable. All of these representations except the trivial one are transitive. Hence K has countably many subgroups of index 5 and none of index 2, 3 or 4.

THEOREM 3.8: *Let K be a finitely presented $\mathbb{Z}$-group. Suppose some representation ρ of K onto S_2 has infinitely many lifts to representations into S_3. Then K has uncountably many representations onto S_n for all $n \geq 3$.*

Proof: The argument is similar to that of the next-to-last paragraph of the proof of Theorem 3.4. By Lemma 3.3, the component of ρ in the lifting of ρ to S_3 is uncountable. The graph of this component contains closed paths p and

$\tilde{p}$ corresponding to ρ and a nontrivial periodic lifting $\tilde{\rho}$. The image $\tilde{\rho}(K)$ must contain (13), (23) or (123), so paths that contain both p and $\tilde{p}$ correspond to representations onto S_3.

Now fix $n \geq 4$. For $4 \leq m \leq n$ we can obtain a periodic representation $\rho^{(m)}$ of K onto the subgroup of S_n consisting of all permutations of $\{1, 2, m\}$ by conjugating $\tilde{\rho}$ by the 2-cycle $(3m)$. Since each of these conjugations fixes ρ, they all leave the component of ρ in the graph of Φ_{S_n} invariant. Any path in this component that contains the closed paths corresponding to ρ, $\tilde{\rho}$ and each $\rho^{(m)}$ must map onto S_n, and there are uncountably many such paths. ∎

COROLLARY 3.9: *Let K be the commutator subgroup of a knot group. If K has infinitely many representations into S_3, then K has uncountably many representations onto S_n for all $n \geq 3$, and hence uncountably many subgroups of every index $n \geq 3$.*

Proof: As we noted at the end of Section 2, the Markov groups $\Phi_{\mathbb{Z}_2}$ and $\Phi_{\mathbb{Z}_3}$ are finite in this case. Hence there are infinitely many representations onto S_3, and infinitely many are lifts of a single representation onto S_2. ∎

4. Additional application to knot groups

Assume that $M \cong R^n/AR^m$ is a finitely generated module over a Noetherian ring R, where the presentation matrix A is an $n \times m$ matrix with entries in R. By adjoining zero columns, we can assume that $m \geq n$. The elementary ideals E_i of A form a sequence of invariants of m. The ideal E_i is generated by the $(n-i) \times (n-i)$ minors of the matrix A. When R is a factorial domain (for example, $\mathbb{Z}[t, t^{-1}]$ or $F[t, t^{-1}]$, where F is a field), each E_i is contained in a unique minimal principal ideal; a generator, which is well defined up to multiplication by units in R, is denoted by $\Delta_i(t)$. If $R = \mathbb{Z}[t, t^{-1}]$, then Δ_i is a polynomial, the ith **characteristic polynomial** of M, and we normalize it so that it has the form $c_0 + c_1 t + \cdots + c_d t^d$, where $c_0 \neq 0$.

Here we are concerned only with the 0th characteristic polynomial. Note that when A is a square matrix, $\Delta_0(t)$ is simply the determinant of A. Such is the case for any knot: If $X = S^3 \setminus k$ is a knot complement, and $\tilde{X}$ is its infinite cyclic cover, then $H_1(\tilde{X}; \mathbb{Z})$ is a finitely generated $\mathbb{Z}[t, t^{-1}]$-module with square presentation matrix. The 0th characteristic polynomial is called the **Alexander polynomial** of k, denoted here by $\Delta(t)$. The Alexander polynomial $\Delta(t)$ of any knot has even degree and satisfies $\Delta(1) = \pm 1$.

For any positive integer r, we can regard M as a finitely generated module over $\mathbb{Z}[s, s^{-1}]$, where $s = t^r$. We can obtain a presentation matrix $A(C_r)$ from A by replacing each polynomial entry $f(t)$ by $f(C_r)$, where C_r is the $r \times r$ companion matrix of the polynomial $t^r - s$:

$$C_r = \begin{pmatrix} 0 & 0 & \cdots & 0 & s \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}.$$

LEMMA 4.1: *Assume that $M \cong R^n/AR^n$ is a finitely generated $R = \mathbb{Z}[t, t^{-1}]$-module with 0th characteristic polynomial $\Delta(t) = c_d \prod(t - \alpha_i)$. Let $s = t^r$, $r \geq 1$. The 0th characteristic polynomial of M regarded as a $\mathbb{Z}[s, s^{-1}]$-module is $\tilde{\Delta}(s) = c_d^r \prod(s - \alpha_i^r)$.*

Proof: Regard $\mathbb{Z}[s, s^{-1}]$ as a subring of $\mathbb{Z}[t, t^{-1}]$, which in turn is a subring of $\mathbb{C}[t, t^{-1}]$. The matrix C_r is similar over $\mathbb{C}[t, t^{-1}]$ to the diagonal matrix $D = \mathrm{Diag}(t, \zeta t, \ldots, \zeta^{r-1} t)$, where ζ is a primitive rth root of unity. Consequently, $\tilde{\Delta}(s) = \mathrm{Det}\, A(D)$, where as above $A(D)$ denotes the matrix obtained from A by replacing each polynomial entry $f(t)$ by $f(D)$. By Theorem 1 of [KSW99] the determinant is equal to $\mathrm{Det}[\Delta(D)]$, which is $\mathrm{Det}[c_d \prod_i (D - \alpha_i I)] = c_d^r \prod_i \prod_j (\zeta^j t - \alpha_i)$. The last product can be rewritten as $c_d^r \prod_i \prod_j (t - \zeta^{-j}\alpha_i)$, which is equal to $c_d^r \prod_i (t^r - \alpha_i^r) = c_d^r \prod_i (s - \alpha_i^r)$. ∎

A polynomial $f(t) \in \mathbb{Z}[t, t^{-1}]$ of degree d is **symmetric** (or **reciprocal**) if $f(t^{-1}) \doteq f(t)$. Here $\doteq$ indicates equality up to multiplication by a unit $\pm t^n$ in $\mathbb{Z}[t, t^{-1}]$. It is well known that the Alexander polynomial $\Delta(t)$ of any knot is a symmetric polynomial of even degree (see [BS03], for example).

COROLLARY 4.2: *Assume the hypotheses of Lemma 4.1. If, moreover, $\Delta(t)$ is symmetric, then so is $\tilde{\Delta}(s)$.*

Proof: A polynomial is symmetric if and only if its set of zeros (with multiplicities) is sent to itself by inversion. The desired conclusion follows immediately. ∎

THEOREM 4.3: *Assume that K is the commutator subgroup of the group of a knot. Then (i) any representation from K onto S_2 lifts to a representation onto S_3; (ii) any representation from K onto $\mathbb{Z}_3$ lifts to a representation onto A_4.*

Proof: (i) Let ρ be a representation from K onto S_2. The (unique) epimorphism $S_3 \to S_2$ fits into a short exact sequence $A = \langle(123)\rangle \to S_3 \to S_2$ that splits, and hence a lifting $\tilde{\rho}$ can be found. By Proposition 3.1, the complete set of liftings ρ_ξ corresponds bijectively to the group of twisted cocycles $C^1(K, \{A\})$. It is not difficult to see ρ_ξ fails to be surjective if and only if ξ is a coboundary. We must prove that $H^1(K, \{A\})$ is nontrivial.

Our proof is topological. We construct a 2-complex X with fundamental group K, and invoke Therorem 4 of [Li95], an application of Shapiro's Lemma, to see that the problem of showing that $H^1(K, \{A\})$ is nontrivial is equivalent to proving that the mod-3 first homology group of a certain cover of X has larger rank than the corresponding homology group of X. The fact that K is the commutator subgroup of a knot ensures that the cover satisfies a type of Poincaré duality, imposing conditions that are sufficient to complete the argument.

Let X be a CW complex with a single vertex and fundamental group K. For any prime p, the homology group $H_1(X; \mathbb{Z}_p)$ is a finitely generated module over the ring $\mathbb{Z}_p[t, t^{-1}]$ of Laurent polynomials with coefficients in $\mathbb{Z}_p$. A square matrix presenting the module can be found, and its determinant $\Delta_0(H_1(X; \mathbb{Z}_p))$ is the Alexander polynomial $\Delta(t)$ of the knot with coefficients reduced modulo p. Alternatively, $H_1(X; \mathbb{Z}_p)$ can be viewed as a finite-dimensional vector space over $\mathbb{Z}_p$. Its dimension is equal to the mod p degree of $\Delta(t)$. All of the above statements hold as well using cohomology.

Let $\pi\colon \tilde{X} \to X$ be the 2-fold cover corresponding to ρ. By Theorem 4 of [Li95], $H^1(K, \{A\}) \cong H^1(\tilde{X}; \mathbb{Z}_3)/\pi^* H^1(X; \mathbb{Z}_3)$. We must prove that the dimension of $H^1(\tilde{X}; \mathbb{Z}_3)$ exceeds that of $H^1(X; \mathbb{Z}_3)$. The Universal Coefficient Theorem implies that the dimension of $H^1(\tilde{X}; \mathbb{Z}_3)$ (resp. $H^1(X; \mathbb{Z}_3)$) is equal to that of $H_1(\tilde{X}; \mathbb{Z}_3)$ (resp. $H_1(X; \mathbb{Z}_3)$). We will work with homology.

Any representation from K to a finite abelian group is periodic [SW99′]. Let r be the period of ρ. Then $H_1(\tilde{X}; \mathbb{Z})$ is a finitely generated module over $\Lambda = \mathbb{Z}[s, s^{-1}]$, where $s = t^r$. In fact ρ induces a homomorphism from the fundamental group of the r-fold cyclic cover X_r of the knot to S_2. (Details can be found in [SW99′].) We can regard $\tilde{X}$ as an infinite cyclic cover of the induced 2-fold cover of X_r. It follows from Blanchfield duality that $\Delta_0(H_1(\tilde{X}; \mathbb{Z}))$ is a symmetric polynomial ([Tu01], Corollary 14.7).

We regard $H_1(X; \mathbb{Z})$ too as a Λ-module with 0th characteristic polynomial $\tilde{\Delta}(s)$. By Lemma 4.1 and Corollary 4.2, $\tilde{\Delta}(s)$ has the same degree as $\Delta(t)$ and is symmetric.

The CW complex $\tilde{X}$ has a 0-skeleton $\tilde{X}^0$ consisting of two vertices. It is convenient to work with the relative homology group $H_1(\tilde{X}, \tilde{X}^0; \mathbb{Z})$; it fits into a short exact sequence

$$0 \to H_1(\tilde{X}; \mathbb{Z}) \to H_1(\tilde{X}, \tilde{X}^0; \mathbb{Z}) \to \Lambda/(s-1) \to 0,$$

from which it follows that

$$\Delta_0(H_1(\tilde{X}, \tilde{X}^0; \mathbb{Z})) \doteq D_0(H_1(\tilde{X}; \mathbb{Z}))(s-1)$$

(see, for example, Lemma 7.2.7 [Ka96]). Consider then the chain complex

$$0 \to C_2(\tilde{X}, \tilde{X}^0; \mathbb{Z}) \xrightarrow{\partial} C_1(\tilde{X}, \tilde{X}^0; \mathbb{Z}) \to 0$$

for the pair $(\tilde{X}, \tilde{X}^0)$. The boundary ∂ can be represented by a matrix of the form

$$T = \begin{pmatrix} A & B \\ B & A \end{pmatrix}.$$

Here A and B are square matrices of the same size. The first half of the columns of T correspond to edges of $\tilde{X}$ that are lifts of edges of X beginning at a fixed vertex of $\tilde{X}^0$; the remaining columns correspond to edges that are lifts beginning at the other vertex.

Row and column operations convert T into

$$T' = \begin{pmatrix} A-B & B \\ 0 & A+B \end{pmatrix}.$$

The matrix $A+B$ is a relation matrix for $H_1(X; \mathbb{Z})$, and consequently its determinant is $\tilde{\Delta}(s)$. Since $\Delta_0(H_1(\tilde{X}; \mathbb{Z}))$ and $\tilde{\Delta}(s)$ are both symmetric polynomials, so is $\det(A-B)$. The latter factors as $(s-1)g(s)$ for some (necessarily symmetric) polynomial $g(s)$, since $s-1$ cannot divide $\tilde{\Delta}(s)$.

Clearly $\det(A-B)$ is congruent modulo 2 to $\det(A+B)$. It follows that $g(s)$ has odd degree. If $g(s)$ (mod 3) is nonzero, then it must have positive degree. Therefore, if $\Delta_0(H_1(\tilde{X}; \mathbb{Z}_3)$ is nonzero, then its degree is larger than that of $\Delta_0(H_1(X; \mathbb{Z}_3)$. Equivalently, if $H_1(\tilde{X}; \mathbb{Z}_3)$ has finite dimension, then its dimension is greater than that of $H_1(X; \mathbb{Z}_3)$.

The proof of (ii) follows a similar line of reasoning as (i). Let ρ be a representation of K onto $\mathbb{Z}_3$. The alternating group A_4 maps onto $\mathbb{Z}_3$ with kernel $A = \langle (12)(34), (13)(24) \rangle \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$, and the resulting short exact sequence splits. Hence a lifting $\tilde{\rho}$: $K \to A_4$ can be found. As in the proof of (i), Proposition 3.1 implies that the complete set of liftings ρ_ξ corresponds bijectively to the group of twisted cocycles $C^1(K, \{A\})$. Again ρ_ξ fails to be surjective if and only if ξ is a coboundary. We must prove that $H^1(K, \{A\})$ is nontrivial.

Let $p\colon \tilde{X} \to X$ be the 3-fold cover corresponding to ρ. By Theorem 4 of [Li95], $H^1(K, \{A\}) \cong H^1(\tilde{X}; \mathbb{Z}_2)/p^*H^1(X; \mathbb{Z}_2)$. Hence it suffices to show that the dimension of $H_1(\tilde{X}; \mathbb{Z}_2)$ exceeds the dimension of $H_1(X; \mathbb{Z}_2)$.

Since the unique vertex of X is covered by 3 vertices in $\tilde{X}$, a short exact homology sequence similar to the one above shows that

$$\Delta_0(H^1(\tilde{X}, \tilde{X}^0; \mathbb{Z}_2)) \doteq D_0(H^1(\tilde{X}; \mathbb{Z}_2))(s-1)^2.$$

The representation $\rho\colon K \to \mathbb{Z}_3$ is necessarily periodic, say of period r. Let $s = t^r$. Then $H_1(\tilde{X}, \tilde{X}^0; \mathbb{Z}_2)$ is a finitely generated $\Lambda = \mathbb{Z}[s, s^{-1}]$-module. We view $H_1(X, X^0; \mathbb{Z}_2)$ likewise as a Λ-module, and again by Blanchfield Duality and Corollary 4.2 its 0th characteristic polynomial $\tilde{\Delta}(s)$ is symmetric of even degree.

As in the proof of (i) we consider the chain complex

$$0 \to C_2(\tilde{X}, \tilde{X}^0; \mathbb{Z}) \xrightarrow{\partial} C_1(\tilde{X}, \tilde{X}^0; \mathbb{Z}) \to 0$$

for the pair $(\tilde{X}, \tilde{X}^0)$. The boundary operator ∂ is represented by a matrix of the form

$$T = \begin{pmatrix} A & B & C \\ B & C & A \\ C & A & B \end{pmatrix},$$

where A, B and C are square matrices of the same size. The first third of the columns of T correspond to edges of $\tilde{X}$ that are lifts of edges of X beginning at a fixed vertex of $\tilde{X}^0$; the second third (resp. last third) correspond to edges that are lifts beginning at the second (resp. third) vertex.

The matrix T is similar in $\mathbb{Z}[\zeta][s, s^{-1}]$ to

$$T' = \begin{pmatrix} A+B+C & 0 & 0 \\ 0 & A+\zeta B+\zeta^2 C & 0 \\ 0 & 0 & A+\zeta^2 B+\zeta C \end{pmatrix},$$

where ζ is a primitive 3rd root of unity. Consequently, $\Delta_0(H_1(\tilde{X}, \tilde{X}^0; \mathbb{Z}))$ is equal to $\mathrm{Det}(R') = \mathrm{Det}(A+B+C)\,\mathrm{Det}(A+\zeta B+\zeta^2 C)\,\mathrm{Det}(A+\zeta^2 B+\zeta C)$, which we write as $\tilde{\Delta}(s)F(s)\bar{F}(s)$. As before $\tilde{\Delta}(s)$ is obtained from the Alexander polynomial of the knot, using Lemma 4.1; $F(s)$ is a polynomial with coefficients in $\mathbb{Z}[\zeta]$, and $\bar{F}(s)$ is the polynomial obtained from $F(s)$ by replacing each coefficient by its conjugate.

As in the proof of (i), $\Delta_0(H_1(\tilde{X}; \mathbb{Z}))$ is a symmetric polynomial, and hence so is $\Delta_0(H_1(\tilde{X}, \tilde{X}^0; \mathbb{Z}))$. Since $\Delta_0(H_1(\tilde{X}, \tilde{X}^0; \mathbb{Z})) = \tilde{\Delta}(s)F(s)\bar{F}(s)$ and

$\tilde{\Delta}(s)$ are symmetric, so is $F(s)\bar{F}(s)$. By the Universal Coefficient Theorem, $\Delta_0(H_1(\tilde{X},\tilde{X}^0;\mathbb{Z}_2))$ is equal to $\tilde{\Delta}(s)F(s)\bar{F}(s)$ with coefficients reduced modulo 2.

The product $F(s)\bar{F}(s)$ has integer coefficients. Since the unique extension of the natural projection $\mathbb{Z} \to \mathbb{Z}_3$ to $\mathbb{Z}[\zeta]$ sends ζ to 1, the product $F(s)\bar{F}(s)$ is congruent modulo 3 to $(\tilde{\Delta}(s))^2$. After suitable normalization, we can write $F(s) = c_d s^d + \cdots + c_1 s + c_0$, where each $c_i \in \mathbb{Z}[\zeta]$. The product $F(s)\bar{F}(s)$ is equal to $c_d\bar{c}_d s^{2d} + \cdots + c_0\bar{c}_0$. If the degree of $F(s)\bar{F}(s)$ decreases when coefficients are reduced modulo 3, then both $c_d\bar{c}_d$ and $c_0\bar{c}_0$ must be divisible by 3. This implies that $c_d\bar{c}_d$ and $c_0\bar{c}_0$ vanish under $\mathbb{Z}[\zeta] \to \mathbb{Z}_3$, and hence both c_d and c_0 also vanish. An induction argument shows that the degees of $F(s)\bar{F}(s)$ and $F(s)\bar{F}(s) \pmod 3$ differ by a multiple of 4. But since $\tilde{\Delta}(s)$ has even degree, the degree of $(\tilde{\Delta}(s))^2$ is a multiple of 4. Hence the degree of $F(s)\bar{F}(s)$ is a multiple of 4, and so the degree of $F(s)$ is even.

Now consider $F(s)\bar{F}(s)$ with coefficients reduced modulo 2. As before, if the reduction causes its degree to decrease, then both $c_d\bar{c}_d$ and $c_0\bar{c}_0$ must be even. This implies that both vanish under the natural projection $\mathbb{Z}[\zeta] \to \mathbb{Z}_2[\zeta]$. Hence $F(s)\bar{F}(s) \pmod 2$ has degree divisible by 4. If it is zero, then $H_1(\tilde{X};\mathbb{Z}_2)$ is infinite, and so its dimension is larger than that of $H_1(X;\mathbb{Z}_2)$. If it is nonzero, then it contains a nontrivial factor other than $(s-1)^2$, and again the dimension of $H_1(\tilde{X};\mathbb{Z}_2)$ is greater than that of $H_1(X;\mathbb{Z}_2)$. ∎

Example 4.4: It is easy to construct examples of general $\mathbb{Z}$-groups for which the conclusions of Theorem 4.3 do not hold. For example, consider the group S_2 presented in the following way:

$$K = \langle a_j \mid a_j^2,\ a_{j+1} = a_j,\ \forall j \rangle.$$

Clearly K admits a (unique) homomorphism onto S_2 that does not lift to S_3. A similar example can be constructed to show that the second conclusion of Theorem 4.3 does not hold for general finitely presented $\mathbb{Z}$-groups.

Nontrivial examples can also be constructed. Consider the $\mathbb{Z}$-group:

$$K = \langle a_j \mid a_{j+1} = a_j^3,\ \forall j \rangle.$$

This group admits a representation ρ onto S_2, mapping each generator a_j to the nontrivial element. The reader can verify that the matrix T in the proof of Theorem 4.3 is

$$\begin{pmatrix} s-2 & -1 \\ -1 & s-2 \end{pmatrix}.$$

The determinant modulo 3 is equal up to a multiplicative unit to $s-1$, and hence the polynomial $g(s)$ is trivial. Consequently, ρ does not lift onto S_3.

For the second part of Theorem 4.3, consider the group

$$K = \langle a_j \mid a_{j+1} = a_j^2 \ \forall j \rangle$$

of Example 2.1. This is the commutator subgroup of a 2-knot group. It admits a representation onto $\mathbb{Z}_3$ mapping each a_{2j} to 2 and each a_{2j+1} to 1. The reader can verify that the matrix T in the proof of Theorem 4.3 is

$$\begin{pmatrix} 1 & -1 & 1 & 0 & 0 & 0 \\ -s & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & -1 & 1 & 0 \\ 0 & 1 & -s & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 & -s & 1 \end{pmatrix}.$$

The determinant modulo 2 is equal up to a multiplicative unit to $(s-1)^2$, and hence $F(s)\bar{F}(s)$ is trivial. Consequently, ρ does not lift onto A_4.

5. Conclusion

The obstruction theory used here has provided new insight into the structure of representation shifts. However, questions remain.

CONJECTURE 5.1: *Dichotomy holds for commutator subgroups of knot groups. That is, for any finite target group Σ, the shift Φ_Σ is either finite or uncountable.*

A more subtle, dynamical conjecture which would imply Conjecture 5.1 is the following.

CONJECTURE 5.2: *For any finite group Σ, periodic points are dense in every representation shift Φ_Σ of a knot commutator subgroup.*

A classical result of Perko [P76] says that every representation of a knot group onto S_3 lifts onto S_4.

CONJECTURE 5.3: *Let K be the commutator subgroup of a knot k. Every representation of K onto S_3 lifts onto S_4.*

Conjecture 5.3 holds for fibered knots, since in such a case K is a nonabelian free group. The general conjecture is equivalent to the assertion that for any surjection ρ: $K \to S_3$, the rank of the mod-2 first homology group of the associated 3-fold dihedral cover $\tilde{Y}$ of the infinite cyclic cover Y of the knot exceeds that of Y.

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